

GLOBAL CONNECTIVITY AND EXPANSION:
LONG CYCLES AND FACTORS IN f -CONNECTED GRAPHSSTEPHAN BRANDT, HAJO BROERSMA, REINHARD DIESTEL,
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Received July 3, 2002

Revised April 1, 2005

Given a function $f: \mathbb{N} \rightarrow \mathbb{R}$, call an n -vertex graph f -connected if separating off k vertices requires the deletion of at least $f(k)$ vertices whenever $k \leq (n - f(k))/2$. This is a common generalization of vertex connectivity (when f is constant) and expansion (when f is linear). We show that an f -connected graph contains a cycle of length linear in n if f is any linear function, contains a 1-factor and a 2-factor if $f(k) \geq 2k + 1$, and contains a Hamilton cycle if $f(k) \geq 2(k + 1)^2$. We conjecture that linear growth of f suffices to imply hamiltonicity.

1. Introduction: the notion of f -connectedness

In his survey on tree-width, tangles and graph minors, Reed [24] starts out by considering the global connectivity properties of a hexagonal grid: he observes that, although any such grid is barely 2-connected, the number of vertices needed to disconnect a *large* part from the rest is large: it grows at least as \sqrt{k} with the size k of that part. Our aim in this paper is to define this notion of global connectivity more formally, to see how it relates to other graph invariants, and in particular to ask what ‘richness’ properties (such as hamiltonicity, or the existence of other global substructures) it can imply.

Given a graph G , let us call a pair (A, B) of proper subsets of $V(G)$ a *separation* of G if $A \cup B = V(G)$ and G has no edge between $A \setminus B$ and $B \setminus A$. Let $f: \mathbb{N} \setminus \{0\} \rightarrow \mathbb{R}$ be a function (usually non-decreasing).

Mathematics Subject Classification (2000): 05C40, 05C38, 05C70, 05C45

Definition 1.1. G is called f -connected if every separation (A, B) of G with $|A \setminus B| \leq |B \setminus A|$ satisfies $|A \cap B| \geq f(|A \setminus B|)$.

Thus, for small k , splitting off k vertices requires a separator of size at least $f(k)$.

For constant functions $f: k \mapsto c \in \mathbb{N}$, f -connectedness reduces to the traditional notion of c -connectedness. Complete graphs are f -connected for every f (because they have no separations), and grids are f -connected for $f: k \mapsto \sqrt{2k}$ (see [6, 25]).

An equivalent way to view f -connectedness is as an expansion property, ie. in terms of an isoperimetric inequality (such as (1) below). Isoperimetric inequalities have been investigated mainly for specific classes of graphs (such as higher-dimensional grids and cubes; see e.g. [6], and [10] for references), while the focus in the study of expanders has largely been to construct (sparse) expanders with certain desired properties [10, 16].

Our aim here differs from both these: rather than trying to determine for which f some specific graphs are f -connected, or to construct f -connected graphs for certain specific f , we shall think of f as given but arbitrary, and then try to relate the corresponding property of f -connectedness to other graph invariants. This has two aspects:

- What properties or substructures of a graph G can we force by assuming that G is f -connected?
- What assumptions will force a graph G to be f -connected or to have an f -connected subgraph (or minor etc)?

We will mainly be concerned here with the first of these aspects. Concerning the second aspect it follows from a result of Komlós and Szemerédi [19, Theorem 2.2] that every graph G has a—possibly very small—subgraph H with at least half the average degree of G which is f -connected for certain sublinear functions f .

Our graph-theoretical notation is that of [13]. In particular, if S is a set of vertices in a graph $G = (V, E)$, we denote by $N_G(S) = N(S)$ the set of neighbours in $G - S$ of vertices in S . Initial, middle and final segments of a path P with vertices x, y are denoted as Py , xPy and xP , respectively (see also [13, p. 7]). Unless otherwise stated, f will always denote an arbitrary non-decreasing $\mathbb{N} \rightarrow \mathbb{R}$ function.

Throughout the paper, we shall use freely the following two reformulations of f -connectedness as an expansion property. For

$$\overline{S} := V \setminus (S \cup N(S)),$$

clearly G is f -connected if and only if

$$|N(S)| \geq f(|S|) \text{ for every } S \subseteq V \text{ with } 0 < |S| \leq |\bar{S}|.$$

(Rephrased like this, *f-edge-connectedness* could be defined correspondingly in terms of edge boundaries.)

Put another way, if $|G| =: n$ and G is f -connected, then a non-empty set $S \subseteq V$ can satisfy $|N(S)| < f(|S|)$ only if $|S| > |\bar{S}| = n - |N(S)| - |S|$, or equivalently if $|N(S)| > n - 2|S|$. Thus, G is f -connected if and only if every $S \subseteq V$ satisfies

$$(1) \quad |N(S)| \geq \min \{f(|S|), n - 2|S| + 1\}.$$

Our paper is organized as follows. We start out in [Section 2](#) with our main results, establishing upper bounds on the growth rate of f needed to force a Hamilton cycle or other long cycle in a given graph. In [Section 3](#) we do the same for 1-factors and 2-factors. [Section 4](#) gives some brief indications of the kind of growth rate of f that can be expected to force standard substructures (such as given subgraphs or minors), and suggests some open problems. In [Section 5](#) we relate f -connectedness formally to expanders, and note a lower bound in terms of eigenvalues for regular graphs. As an application of our main results we deduce the existence of long cycles and factors in Ramanujan graphs.

2. Forcing long cycles

It is well known that random graphs contain long cycles, indeed Hamilton cycles, even when they are relatively sparse. On the other hand, they are well-expanding. Our aim in this section is to relate these two properties directly: we shall prove that linear f -connectedness always forces a cycle of length linear in the order of the graph, and that quadratic f -connectedness can force it to be hamiltonian.

Theorem 2.1. *If $f(k) \geq 2(k+1)^2$ for all $k \in \mathbb{N}$, then every f -connected graph of order $n \geq 3$ is hamiltonian.*

We do not believe that quadratic growth of f is needed in [Theorem 2.1](#), but expect that some linear growth will do:

Conjecture 2.2. *There exists a function $f(k) = O(k)$ such that every f -connected graph of order $n \geq 3$ is hamiltonian.*

We shall see later (Section 4) that Conjecture 2.2 follows from Chvátal's toughness conjecture [11].

Since bipartite graphs $K_{n,n-1}$ are not hamiltonian but f -connected for $f: k \mapsto 2k$ when n is odd, the strongest possible version of Conjecture 2.2 would be the following:

Conjecture 2.3. If $f(k) \geq 2k+1$ for all $k \in \mathbb{N}$, then every f -connected graph of order $n \geq 3$ is hamiltonian.

Although we cannot prove either of these conjectures, we can show that an f -connected graph with $f(k) \geq 2k$ contains a cycle of length at least $3n/4$. This is the case $c=2$ of the following more general result:

Theorem 2.4. Let $f(k) \geq ck$ for some real number $c \geq 2$ and all $k \in \mathbb{N}$, and let G be an f -connected graph of order $n \geq 3$. Then G has a cycle of length at least $\frac{c+1}{c+2}n$.

For $c=2$, this improves the lower bound of $5n/9$ obtained by Rempel [25].

We shall prove Theorems 2.1 and 2.4 by the same method. This method does not allow an extension of Theorem 2.4 to $c < 2$. For $c < 2$ we can still show that f -connected graphs have cycles of linear length, but only with a smaller multiplicative constant:

Theorem 2.5. Let $f(k) \geq ck$ for some real number $c \in (0, 2)$ and all $k \in \mathbb{N}$, and let G be an f -connected graph of order $n > 1 + \frac{3}{c}$. Then G has a cycle of length at least $(\frac{c}{c+2})^2 n$.

We remark that the lower bound on n in Theorem 2.5 is necessary: there can be trees of order up to $1 + \frac{3}{c}$ that are f -connected with $f(k) = ck$. See the remark following Lemma 2.10.

In our proofs we shall use the following tool due to Pósa [23]. Let P be a path in a graph G , say from u and v . Given a vertex $x \in P$, we write x^- for the vertex preceding x on P , and x^+ for the vertex following x on P (whenever these exist). Similarly for $X \subseteq V(P)$ we put

$$X^- := \{x^- \mid x \in X\} \quad \text{and} \quad X^+ := \{x^+ \mid x \in X\}.$$

If $x \in P - u$ is a neighbour of u in G , then $P - xx^- + ux$ (which is a path in G with vertex set $V(P)$) is said to have been obtained from P by an *elementary exchange* fixing v (Figure 1). A path obtained from P by a (possibly empty) sequence of elementary exchanges fixing v is a path *derived* from P . The set of starting vertices of paths derived from P , including u , will be denoted

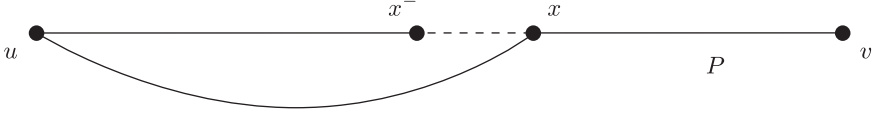


Figure 1. An elementary exchange applied to the path P

by $S(P)$. As all paths derived from P have the same vertex set as P , we have $S(P) \subseteq V(P)$.

The following lemma is a one-sided variant of Pósa's Lemma [23]; see also [7, Lemma 6.3].

Lemma 2.6. *Let G be a graph, let $P = u \dots v$ be a longest path in G , and put $S := S(P)$. Then $N_G(S) \subseteq S^- \cup S^+$.*

Proof. Let $x \in S$ and $y \in N(x)$ be given; we show that $y \in S^- \cup S \cup S^+$. As $x \in S$ there is a path $Q = x \dots v$ derived from P . Then $y \in V(Q) = V(P)$, because Q is a longest path; let z denote the predecessor of y on Q .

Suppose that $y \notin S^- \cup S \cup S^+$. Then each of the (one or two) edges $e \in P$ at y lies on every path derived from P (and in particular on Q), because in any elementary exchange in which e is first deleted, its two ends (including y) would have become members of S and of $S^- \cup S^+$, respectively. Hence $z \in \{y^-, y^+\}$. But $Q + xy - yz$ is obtained from Q by an elementary exchange, which puts z in S and y in $S^- \cup S^+$. ■

Lemma 2.7. *Let G be a graph, let $P = u \dots v$ be a longest path in G , and put $S := S(P)$. Then G has a cycle containing $S \cup N(S)$.*

Proof. Let y be the last vertex of P in $N(S)$. Then all the vertices from $S \cup N(S)$ lie on Py , because any vertex of yP in S would differ from v and hence have its successor on P in $N(S)$. Let $x \in S$ be a neighbour of y in G , and let $Q = x \dots v$ be derived from P . As in the proof of Lemma 2.6, all the edges of yP are still edges of Q , so $yQ = yP$. Thus $S \cup N(S) \subseteq V(Py) = V(Qy)$, and Qyx is a cycle in G . ■

Proof of Theorem 2.4. Let $P = u \dots v$ be a longest path in G , and put $S := S(P)$.

Suppose first that $|S| \leq n/(c+2)$. Then

$$|N(S)| \underset{(1)}{\geq} \min\{c|S|, n - 2|S| + 1\} \geq c|S|,$$

which by Lemma 2.6 implies the contradiction of

$$3|S| - 1 \geq |S^+ \cup S \cup S^-| \geq |S \cup N(S)| \geq |S| + c|S| \geq 3|S|.$$

(The -1 comes from the fact that $u \in S$ while u^- does not exist.)

We may therefore assume that $|S| > n/(c+2)$. Choose a subset $S' \subseteq S$ with

$$|S'| = \left\lceil \frac{n}{c+2} \right\rceil.$$

Then

$$|S' \cup N(S')| \underset{(1)}{\geq} \min\{(c+1)|S'|, n - |S'| + 1\} \geq \frac{c+1}{c+2} n.$$

Now the cycle from [Lemma 2.7](#), which contains $S \cup N(S) \supseteq S' \cup N(S')$, has the desired length. \blacksquare

Proof of Theorem 2.1. Let G be an f -connected graph with $|G| =: n \geq 3$ and $f(k) \geq 2(k+1)^2$ for all $k \in \mathbb{N}$. We first show that G contains a longest path $P = u \dots v$ with

$$(2) \quad \sqrt{2n} - 2 < d(u) \leq d(v).$$

Choose $P = u \dots v$ among all the longest paths in G with $d(u) \leq d(v)$ so that $d(u)$ is maximum, and subject to this with $d(v)$ maximum. Put $S := S(P)$. By [Lemma 2.6](#) and (1), we have

$$2|S| \geq |S^- \cup S^+| \geq |N(S)| \geq \min\{2(|S| + 1)^2, n - 2|S| + 1\}$$

and hence $2|S| \geq n - 2|S| + 1$, giving

$$|S| \geq \lceil (n+1)/4 \rceil \geq \lfloor \sqrt{n/2} \rfloor.$$

Consider a subset $S' \subseteq S$ with $|S'| = \lfloor \sqrt{n/2} \rfloor$. By (1), we have

$$|N(S')| \geq \min\{2(|S'| + 1)^2, n - 2|S'| + 1\}.$$

As $2(|S'| + 1)^2 > n$ by the choice of S' , this yields

$$|N(S')| \geq n - 2|S'| + 1 > n - \sqrt{2n},$$

so some vertex $s \in S'$ has degree at least

$$\frac{|N(S')|}{|S'|} > \frac{n - \sqrt{2n}}{\sqrt{n/2}} = \sqrt{2n} - 2.$$

As $s \in S$, there is a longest path $Q = s \dots v$ in G . Thus if (2) fails and hence $d(u) < d(s)$, then the existence of Q contradicts the choice of P . This completes the proof that P satisfies (2).

We next show that G contains a cycle C with $V(C) = V(P)$. Since G is connected and P is a longest path, this cycle must then be a Hamilton cycle. If $uv \in E(G)$ then $C := Pvu$ will do, so let us assume that $uv \notin E(G)$.

Let w be the first vertex on P such that some $x \in \{u, v\}$ has at least half its neighbours on Pw . Put $P_1 := Pw$ and $N_1 := N(x) \cap V(P_1)$, and let $S_1 := N_1^-$. Note that $u \notin N_1$ since $uv \notin E(G)$, so

$$|S_1| = |N_1| \geq d(x)/2 > \sqrt{n/2} - 1$$

by (2). Now let y be such that $\{x, y\} = \{u, v\}$, put $P_2 := wP$ and $N_2 := N(y) \cap V(P_2)$, and let $S_2 := N_2^+$. Note that, by the choice of w , at least half the neighbours of y lie in N_2 . Thus as before, $uv \notin E(G)$ and (2) imply that

$$|S_2| = |N_2| \geq d(y)/2 > \sqrt{n/2} - 1.$$

For $i = 1, 2$ we have

$$|N(S_i)| \geq \min \{2(|S_i| + 1)^2, n - 2|S_i| + 1\}$$

by (1). Since $2(|S_i| + 1)^2 > n$, this means that $|N(S_i)| > n - 2|S_i|$. Applying this with S_i denoting the smaller of the sets S_1 and S_2 , we obtain

$$|N(S_i)| > n - 2|S_i| \geq n - (|S_1| + |S_2|).$$

So this S_i must have a neighbour in the other S_j , ie. G contains an edge e between S_1 and S_2 . Figure 2 shows how this edge can be used to obtain the desired cycle C in each of the two cases of either $x = u$ and $y = v$ or $x = v$ and $y = u$. ■

We finally turn to the proof of Theorem 2.5. We start with another Lemma.

Lemma 2.8. *Let T be a tree of order at least 3, and $r \in T$ a vertex. Then T contains a path $P = r \dots s$ where s is not a leaf, such that $V(T - P)$ is a union of two disjoint sets A and B , possibly empty, whose sizes differ by at most 1 and which are not joined by an edge of T .*

Proof. Let $P = r \dots s$ and A, B be as above ignoring the size condition, but with $|A|$ and $|B|$ as equal as possible. Suppose that $|A| \leq |B| - 2$. If s is adjacent to a leaf in B , then moving this leaf to A decreases the difference between $|A|$ and $|B|$, a contradiction. If s has a neighbour $t \in B$ that is not a leaf, the extended path Pst with $V(T - Pst)$ split into A and $B \setminus \{t\}$ yields a contradiction. So s has no neighbour in B . Then $s \neq r$, and if $P = rs$ then r is not a leaf. So $P' := P - s$ is a path not ending in a leaf, and $V(T - P')$ splits into the sets $A \cup \{s\}$ and B (a contradiction). ■

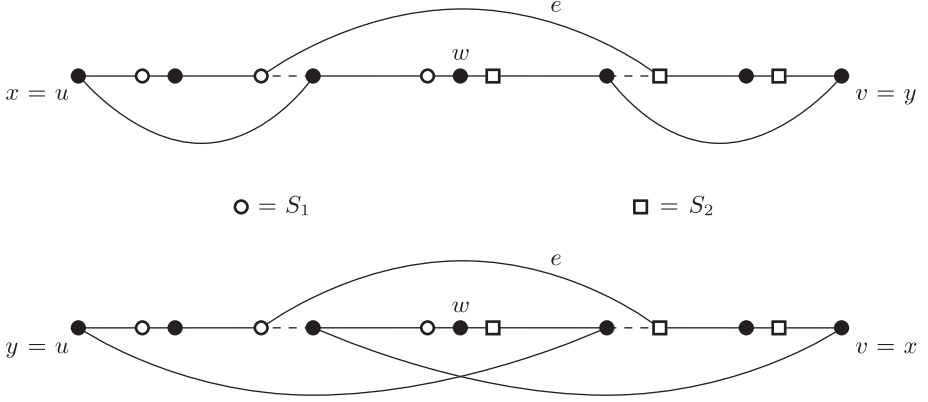


Figure 2. Finding the cycle C in the proof of [Theorem 2.1](#)

Lemma 2.9. Suppose that $f(k) \geq ck$ for some real number $c > 0$ and all $k \in \mathbb{N}$, and let G be an f -connected graph of order $n \geq 3$. Then G contains a path of order at least $\frac{c}{c+2}n$.

Proof. As $f(k) > 0$ for all $k > 0$, we know that G is connected. Let $T \subseteq G$ be a DFS-tree (a *normal* spanning tree in [13]), with root r say. By [Lemma 2.8](#) there is a path $P = r \dots s$ in T such that s is not a leaf and $V(T - P)$ is a union of disjoint sets A, B that are not joined by an edge of T and whose sizes $|A| \leq |B|$ differ by at most 1. Since T is normal in G , we have $N_G(A) \subseteq V(P)$: as P is down-closed in the tree-order of T associated with r , every component of $T - P$ and hence A is up-closed, so any neighbour outside A of a vertex $a \in A$ lies below a and therefore not in B (since B too is up-closed).

As s is not a leaf, P is not a longest path in G . Thus if p denotes the order of a longest path in G , then

$$|A| \geq \frac{1}{2}(n - |P| - 1) \geq \frac{1}{2}(n - (p - 1) - 1) = \frac{1}{2}(n - p).$$

Since $N(A) \subseteq V(P)$, this implies

$$p > |P| \geq |N(A)| \geq c|A| \geq \frac{1}{2}c(n - p),$$

giving $p > \frac{c}{c+2}n$ as desired. ■

Lemma 2.10. Suppose that $f(k) \geq ck$ for some real number $c > 0$ and all $k \in \mathbb{N}$, and let G be an f -connected graph of order $n > 1 + \frac{3}{c}$. If $n \geq 3$, then G contains a cycle.

Proof. As $f(k) > 0$ for all $k > 0$, we know that G is connected. Suppose the lemma fails, ie. that G is a tree. Orient every edge of G towards where more

vertices of G lie, breaking ties arbitrarily, and let v be a sink. We claim that $V(G-v)$ is a union of disjoint sets A and B such that $\frac{1}{3}(n-1) \leq |A| \leq |B|$.

To show this, let us collect components of $G-v$ greedily (large sizes first) until we have collected at least $\frac{1}{3}(n-1)$ vertices, and call this set of vertices C . Then $\frac{1}{3}(n-1) \leq |C| < \frac{2}{3}(n-1)$, and we let A be the smaller of the sets C and $V(G-v) \setminus C$, and B the larger.

Now $|N(A)| = |\{v\}| = 1$, so for $k := |A|$ we have $ck \leq f(k) \leq 1$ and hence $\frac{1}{3}(n-1) \leq k \leq 1/c$. Thus $n \leq 1 + \frac{3}{c}$, contrary to our assumption. ■

We remark that the requirement of $n > 1 + \frac{3}{c}$ in [Lemma 2.10](#) is indeed necessary if we want linear f -connectedness to force the existence of a cycle. Indeed, for every $c > 0$ with $1/c$ integral there is a tree of order $n = 1 + \frac{3}{c}$ that is f -connected with $f(k) = ck$: take any tree with a vertex of degree 3 whose deletion yields three subtrees of order $1/c$.

Lemma 2.11. *Suppose that $f(k) \geq ck$ for some real number $c > 0$ and all $k \in \mathbb{N}$, and let G be an f -connected graph of order $n > 1 + \frac{3}{c}$. Let p denote the order of a longest path in G . If $n \geq 3$, then G contains a cycle of length at least $\frac{c}{c+2}p$.*

Proof. Let P be a longest path in G . Let S_1 denote the set of the first $\lfloor \frac{p}{c+2} + 1 \rfloor$ vertices of P , let S_2 denote the set of its last $\lfloor \frac{p}{c+2} + 1 \rfloor$ vertices, and put $P' := P - S_1 - S_2$.

If $P' = \emptyset$, then $p \leq 2 \lfloor \frac{p}{c+2} + 1 \rfloor \leq \frac{2p}{c+2} + 2$, and hence $\frac{c}{c+2}p \leq 2$. So the length condition on our desired cycle is void, and an arbitrary cycle is provided by [Lemma 2.10](#).

We may therefore assume that $|P'| \geq 1$; then $S_1 \cap S_2 = \emptyset$. If G contains an S_1 – S_2 path $Q = s_1 \dots s_2$ avoiding P' , then $s_1 P s_2 Q s_1$ is a cycle of length at least

$$|P'| + 2 = p - 2 \left\lfloor \frac{p}{c+2} + 1 \right\rfloor + 2 \geq p - \frac{2p}{c+2} = \frac{c}{c+2}p.$$

If there is no such path Q then $V(P')$ separates S_1 from S_2 , so G has a separation (A, B) with $S_1 \subseteq A \setminus B$ and $S_2 \subseteq B \setminus A$ and $V(P') = A \cap B$. But if $|A \setminus B| \leq |B \setminus A|$, say, then the fact that

$$|A \setminus B| \geq |S_1| > \frac{p}{c+2}$$

while

$$|A \cap B| = |P'| = p - 2 \left\lfloor \frac{p}{c+2} + 1 \right\rfloor < p - \frac{2p}{c+2} = c \frac{p}{c+2}$$

contradicts the f -connectedness of G with $f(k) \geq ck$. ■

Now we are ready to complete the proof of [Theorem 2.5](#).

Proof of Theorem 2.5. Our assumptions about c and n imply that $n \geq 3$. By Lemma 2.9, the graph G contains a path of order at least $\frac{c}{c+2}n$. By Lemma 2.11, this implies the existence of a cycle in G of length at least $(\frac{c}{c+2})^2 n$. ■

3. Forcing factors

We have shown that graphs satisfying the premise of Conjecture 2.3 contain if not a Hamilton cycle then at least a very long cycle. Another way to relax the statement of Conjecture 2.3 is to ask for a 2-factor instead of a Hamilton cycle (which is just a connected 2-factor).

It turns out that the bound of Conjecture 2.3 does indeed imply the existence of a 2-factor:

Theorem 3.1. *If $f(k) \geq 2k + 1$ for all $k \in \mathbb{N}$, then every f -connected graph G of order at least 3 has a 2-factor.*

Slightly smaller f -connectedness still forces a 1-factor:

Theorem 3.2. *If $f(k) \geq 2k$ for all $k \in \mathbb{N}$, then every f -connected graph G of even order contains a 1-factor.*

In contrast to our results for cycles, these two statements are easily seen to be best possible. Indeed, $K_{n,n-1}$ has no 2-factor but is f -connected for $f: k \mapsto 2k$ when n is odd, while $K_{n,n-2}$ has no 1-factor but has even order and is f -connected for $f: k \mapsto 2k - 1$ when n is odd.

Proof of Theorem 3.2. If G has no 1-factor then, by Tutte's 1-factor theorem [13], G has a set S of vertices such that $|S| < k$ for the number k of odd components of $G - S$. Since G has even order, k and $|S|$ have the same parity, so $k \geq |S| + 2$.

Let A be the union of the vertex sets of the $\lfloor k/2 \rfloor$ smallest odd components of $G - S$, and let B be the set of all the other vertices of $G - S$. Then $|A| \leq |B|$ but

$$|S| \leq k - 2 < 2\lfloor k/2 \rfloor \leq 2|A|,$$

contradicting the f -connectedness assumed for G . ■

For 2-factors we use Tutte's f -factor theorem [26, 27] instead of his 1-factor theorem. (Readable secondary sources for the f -factor theorem include [5] and [21].)

Proof of Theorem 3.1. By Tutte's f -factor theorem, $G = (V, E)$ has a 2-factor if and only if for all disjoint subsets $A, B \subseteq V$

$$\Theta_G(A, B) := 2|A| + \sum_{v \in B} d_{G-A}(v) - 2|B| - \text{odd}_G(A, B) \geq 0.$$

Here, $\text{odd}_G(A, B)$ denotes the number of *odd components* with respect to (A, B) , which are those components D of $G - A - B$ such that G has an odd number of edges between B and D .

Let G be a counterexample to the theorem with as many edges as possible. Since $\Theta_G(A, B)$ is clearly always even, there are disjoint $A, B \subseteq V$ such that

$$(3) \quad \Theta_G(A, B) \leq -2.$$

Put $C := G - A - B$.

If $b \in B$ is adjacent to some other vertex in B , then $\Theta_G(A, B \setminus \{b\}) \leq \Theta_G(A, B)$. Replacing B with a subset if necessary, we may therefore assume that B is an independent set of vertices.

Let us call a component of C *even* if it is not odd with respect to (A, B) . Adding an edge between a vertex in A and any other vertex, or inside a component of C , or between two even components, or between an odd and an even component, or between B and an even component will not affect $\Theta(A, B)$. By maximality of G it follows that every vertex $a \in A$ is adjacent to every vertex $v \neq a$ of G , that all the components of C are complete, that there is at most one even component, that there is no even component if there is an odd component, and that B is completely joined to every even component.

Let us assume first that there is no odd component. Then two vertices $x, y \in V$ are adjacent if and only if $\{x, y\} \not\subseteq B$, so by (3) we have

$$2|B| \geq 2|A| + 2 + p,$$

where p denotes the number of edges between B and C . If $|B| \leq 1$ then this gives $|A| = p = 0$ and hence $|G| \leq 1$, contrary to our assumption. If $|B| \geq 2$ then $p \geq 2|C|$, and hence

$$|B| \geq |A| + 1 + |C| = |V \setminus B| + 1.$$

Choose a set $X \subseteq B$ of size $\lfloor |B|/2 \rfloor$. Then

$$|N(X)| = |V \setminus B| \leq |B| - 1 \leq 2|X|,$$

contradicting our assumption that G is f -connected with $f(k) \geq 2k + 1$.

We may therefore assume that there is at least one odd component and hence no even component. Let us inductively mark some edges between B and C , two at a time, according to the following rules:

- (1) As often as possible, mark two edges that are both unmarked and have a common end in C . These pairs of edges will be called \vee -pairs.
- (2) Then, as often as possible, mark any two unmarked edges whose ends in C lie in the same component of C .

Then for each component D of C exactly one B – D edge remains unmarked. Thus if p denotes the number of marked pairs, (3) translates to

$$(4) \quad |B| \geq |A| + 1 + p.$$

Write B as a union of two disjoint subsets B_1 and B_2 , of sizes $|B_1| \leq |B_2|$ as equal as possible. Consider a vertex $x \in C$. Since step (1) of our marking algorithm was performed as often as possible, x has at most one neighbour $y \in B$ such that the edge xy lies in no \vee -pair. For every component D of C , let D_1 denote the set of all $x \in D$ for which such a neighbour y exists and lies in B_1 , and define D_2 correspondingly. Then

$$D_1 \cap D_2 = \emptyset \quad \text{and} \quad |D_1 \cup D_2| = 2q + 1,$$

where q denotes the number of pairs of B – D edges marked in (2). (The $+1$ comes from the unmarked B – D edge.) Thus, $|D_1| \neq |D_2|$.

Let us define a partial colouring of V , as follows. Begin by giving the vertices in B_1 colour 1, and those in B_2 colour 2. Now consider separately all the components D of C .

Suppose first that $|D_1| \geq |D_2|$; then $|D_2| \leq q$. Colour with colour 1 all the vertices in D_1 that are not adjacent to B_2 . Then any vertex in D that remains uncoloured either lies in D_2 or is a vertex $x \in D_1$ with a neighbour $y \in B_2$, in which case xy lies in some \vee -pair (because $x \in D_1$). Since $|D_2| \leq q$ it follows that the number of vertices in D that remain uncoloured is at most the total number of marked pairs of B – D edges. On the other hand if $|D_1| < |D_2|$, colour with colour 2 all the vertices in D_2 that are not adjacent to B_1 . As before, the number of vertices in D that remain uncoloured is at most the total number of marked pairs of B – D edges. In total, at most $|A| + p$ vertices in V remain uncoloured, and vertices of distinct colours are nonadjacent.

Let $V_1, V_2 \subseteq V$ be the two colour classes, assume that $|V_1| \leq |V_2|$, and let U be the set of uncoloured vertices. Then $|V_1| \geq \lfloor |B|/2 \rfloor$ (because V_1 contains either B_1 or B_2), and U separates V_1 from V_2 in G . So by f -connectedness we have

$$|A| + p \geq |U| \geq 2|V_1| + 1 \geq |B|,$$

contradicting (4). ■

4. Other substructures forced by f -connectedness

In this section we list some observations on what kind of substructures other than long cycles and factors an assumption of f -connectedness might force in a graph. The aim is more to help the reader find his bearings with the growth rates of f involved here than to prove any further results. So we omit most of the proofs; these, and further details, can be found in the preprint version of this paper [8]. We also mention some open problems.

Since the idea behind the concept of f -connectedness is that it should be a global property, we may ask in particular to what extent it can force substructures that may also occur as global phenomena, such as high chromatic number or large complete minors.

As for chromatic number, note that f -connected graphs for fast growing f cannot have large independent sets of vertices, because dividing such a set into two halves defines a separation. This is made explicit in the proof of the following proposition, which we include as a ‘minimal’ example how f -connectedness can be applied as a tool.

Proposition 4.1. *Let $t \in \mathbb{N}$, and let G be an f -connected graph of order n . If $f(k) \geq 2t(k+1)$ for all $k \in \mathbb{N}$, then $\chi(G) \geq t+2$.*

Proof. Assume that $\chi(G) \leq t+1$. Any $(t+1)$ -colouring of G has a colour class S with at least $\frac{n}{t+1} \geq 2k$ vertices. Choose $S' \subseteq S$ with $|S'| = k$. Then $N(S') \subseteq V(G) \setminus S$, so

$$|N(S')| \leq \left(1 - \frac{1}{t+1}\right)n = 2t \frac{n}{2(t+1)} < 2t(k+1).$$

This contradicts the f -connectedness of G , because $S \setminus S' \subseteq \overline{S'}$ and hence $|S'| \leq |S \setminus S'| \leq |\overline{S'}|$. ■

Complete $(t+1)$ -partite graphs show that [Proposition 4.1](#) cannot be much improved for arbitrary graphs [8].

Friedman and Pippenger [17] proved (a result implying) that, given $d, t \in \mathbb{N}$, every graph satisfying $|N(S)| \geq (d+1)|S|$ for all sets S of at most $2t-2$ vertices contains every tree of maximum degree at most d and order at most t as a subgraph. In terms of f -connectedness, this implies the following:

Proposition 4.2 ([8]). *Let $d \in \mathbb{N}$, and let G be an f -connected graph of order n with $f(k) \geq (d+1)k$ for all $k \in \mathbb{N}$. Then G contains every tree of maximum degree at most d and order $t \leq \frac{n}{2d+6} + 1$.*

The occurrence of a given subgraph will hardly count as a global phenomenon. But for the record let us estimate the growth rate of f needed to force a K_r subgraph by f -connectedness. Exactly as in [Proposition 4.1](#) one shows that if $G \not\supseteq K_r$ then, by Ramsey's theorem, G has an independent set too large for it to be f -connected with $f(k) \geq R(r, 2k+2) - 2k$. With the usual Erdős-Szekeres [\[15\]](#) bound for the Ramsey number $R(r, s)$, this yields the following:

Proposition 4.3 ([\[8\]](#)). *For every $r \in \mathbb{N}$ there is a polynomial $p_r(k) = O(k^{r-1})$ such that every p_r -connected graph has a K_r subgraph.*

For $r=3$ we can do a little better by Kim's theorem [\[18\]](#), which implies that $p_3(k) = O(k^2/\log k)$. But note that even this bound for p_3 is hardly smaller than $2(k+1)^2$, which by [Theorem 2.1](#) already forces a Hamilton cycle.

In contrast to complete subgraphs, dense minors can occur in sparse graphs as a global phenomenon. It would thus seem that forcing large complete minors (say) might be a likely feat of f -connectedness which other invariants cannot as easily achieve. However, this is not the case. As grids are about $\sqrt{\cdot}$ -connected but have no K_5 minor, no function $f(k) = o(k^{1/2})$ could achieve this. On the other hand, a theorem of Alon, Seymour and Thomas [\[2\]](#) implies that if H is any fixed graph and $f(k) = \omega(k^{1/2})$, then all large enough f -connected graphs have an H minor. The question that remains is what happens when $f(k)$ has order exactly $k^{1/2}$:

Problem 4.4. Is there a function $f(k) = O(k^{1/2})$ such that, given any graph H , every large enough f -connected graph has an H minor?

We expect the answer to be negative, ie. that for every $c \in \mathbb{R}$ there exist an $r \in \mathbb{N}$ and arbitrarily large $c\sqrt{\cdot}$ -connected graphs that have no K_r minor.

More positively, one might speculate whether all graphs that are about $\sqrt{\cdot}$ -connected and have no f -connected subgraph for faster-growing functions f look like grids:

Problem 4.5. Given constants $0 < c < d$ and $s \in \mathbb{N}$, is every $c\sqrt{\cdot}$ -connected graph that has no $d\sqrt{\cdot}$ -connected subgraph of order at least s grid-like in some well-specified sense (depending on c , d and s)?

See [\[14\]](#) for some ideas on how grid-likeness might be specified.

A graph that is highly connected in some global sense ought to have large tree-width (for a definition of tree-width see e.g. [\[13, Chapter 12.3\]](#)). Our next observation confirms this for f -connectedness.

Proposition 4.6 ([8]). *Let f, g be two non-decreasing $\mathbb{R}^+ \rightarrow \mathbb{R}$ functions such that $g(n) < n/4$ and $f \geq 4g$ and g is concave for sufficiently large n . Then every f -connected graph of large enough order n has tree-width at least $g(n)$.*

A graph G is called t -tough if for every separating set $S \subseteq V(G)$ the graph $G-S$ has at most $|S|/t$ components. Chvátal [11] conjectured that there is a constant t such that every t -tough graph is hamiltonian. It is known [4] that t cannot be less than $9/4$. Our next result implies that if Chvátal's conjecture is true, with $t=t_0$ say, then every f -connected graph with $f(k) \geq 2t_0(k+1)$ is hamiltonian:

Proposition 4.7. *Let $t \in \mathbb{N}$. If $f(k) \geq 2t(k+1)$ for all $k \in \mathbb{N}$, then every f -connected graph is t -tough.*

Proof. Suppose G is not t -tough, and let S be a separating set of k vertices whose deletion results in more than k/t components. Let A be the union of S and the smallest $\lfloor k/2t \rfloor$ of those components. Then taking B to be the union of S and the other components makes (A, B) into a separation in which A is the smaller side and $|A \setminus B| \geq \lfloor k/2t \rfloor$. Since $f(\lfloor k/2t \rfloor) > k$, the fact that $|A \cap B| = |S| = k$ means that G is not f -connected. ■

Conjecture 2.2 can thus be viewed as a relaxation of Chvátal's conjecture.

5. Expanding graphs and f -connectedness

The expansion of a graph can be estimated in terms of the eigenvalues of certain matrices associated with this graph. In this section we relate f -connectedness for a linear function f to expansion and we determine a linear lower bound for the function f for a regular graph in terms of the eigenvalues of its adjacency matrix. Combined with our results from Sections 2 and 3 this will be applied to show the existence of cycles and factors in so-called Ramanujan graphs.

There are several concepts measuring the expansion of a graph. One of the earliest of them is that of an (n, d, c) -expander. Let $G = (V, E)$ be a d -regular graph of order n , and let $c \in (0, 1]$. G is called an (n, d, c) -expander [3] if for every set $X \subseteq V(G)$ with $|X| \leq \frac{n}{2}$ we have $|N(X)| \geq c|X|$. Since $|A| \leq \frac{n}{2}$ if $|A| \leq |\bar{A}|$, every (n, d, c) -expander is f -connected for $f(k) = ck$. Conversely, we have the following:

Proposition 5.1. *If G is d -regular and f -connected with $f(k) \geq ck$. Then G is an (n, d, c') -expander with $c' = \frac{c}{c+1}$.*

Proof. Let $A \subseteq V(G)$ with $|A| \leq \frac{n}{2}$ be given. If $|A| \leq |\overline{A}|$, then $|N(A)| \geq c|A| \geq c'|A|$ as required. If not, then $|\overline{A}| < |A|$ and hence $|N(\overline{A})| \geq c|\overline{A}|$. Since $N(\overline{A}) \subseteq N(A)$ and $n - |A| \geq |A|$, we obtain

$$|N(A)| \geq |N(\overline{A})| \geq c|\overline{A}| = c(n - |A| - |N(A)|) \geq c(|A| - |N(A)|)$$

and hence $|N(A)| \geq \frac{c}{c+1}|A| = c'|A|$. ■

Let $d = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the spectrum of the adjacency matrix of G , and let

$$\lambda := \max\{|\lambda_2|, |\lambda_n|\}$$

be the largest absolute value of an eigenvalue other than λ_1 .

Lemma 5.2. *Every d -regular graph G is f -connected with $f(k) \geq (\frac{d}{\lambda} - 1)k$.*

In our proof of [Lemma 5.2](#) we will use the following inequality for d -regular graphs [[3](#), Theorem 9.2.4]: for all $S \subseteq V$ and $s := |S|$,

$$(5) \quad \sum_{v \in V} \left(|N(v) \cap S| - \frac{ds}{n} \right)^2 \leq \lambda^2 \frac{s(n-s)}{n}.$$

Note that the left hand side is a measure for the total deviation of the number of neighbours in S of the individual vertices from their average value.

Proof of Lemma 5.2. Choose a set $S \subseteq V$ with $s := |S| \leq |\overline{S}| =: \overline{s}$ such that

$$c := \frac{|N(S)|}{|S|}$$

is minimum. Then G is f -connected with $f(k) = ck$, so all we have to show is that $c \geq \frac{d}{\lambda} - 1$. Note that

$$(6) \quad \frac{n-s}{\overline{s}} = \frac{|N(S)|}{|\overline{S}|} + 1 \leq \frac{|N(S)|}{|S|} + 1 = c + 1 = \frac{|S| + |N(S)|}{|S|} = \frac{n - \overline{s}}{s}.$$

By definition of \overline{S} we have $N(v) \cap S = \emptyset$ for every $v \in \overline{S}$. Double counting gives

$$\sum_{v \in V \setminus \overline{S}} |N(v) \cap S| = 2|E(G[S])| + |E(S, N(S))| = \sum_{v \in S} d(v) = ds.$$

Thus,

$$\begin{aligned}
 \sum_{v \in V} \left(|N(v) \cap S| - \frac{ds}{n} \right)^2 &= \sum_{v \in \bar{S}} \left(|N(v) \cap S| - \frac{ds}{n} \right)^2 \\
 &\quad + \sum_{v \in V \setminus \bar{S}} \left(|N(v) \cap S| - \frac{ds}{n} \right)^2 \\
 &\geq \bar{s} \left(\frac{ds}{n} \right)^2 + (n - \bar{s}) \left(\frac{ds}{n - \bar{s}} - \frac{ds}{n} \right)^2 \\
 &= \left(\frac{ds}{n} \right)^2 \frac{n\bar{s}}{n - \bar{s}},
 \end{aligned}$$

where the inequality follows from the fact that a sum of squares of terms with constant sum is minimized if all summands are equal. Combining this with (5) and using (6) we obtain

$$\left(\frac{d}{\lambda} \right)^2 \leq \frac{n - \bar{s}}{s} \cdot \frac{n - s}{\bar{s}} \leq (c + 1)^2$$

and hence $c \geq \frac{d}{\lambda} - 1$, as desired. ■

Recall that the eigenvalues λ_i of a d -regular graph satisfy

$$d = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq -d.$$

The graph is bipartite if and only if $\lambda_n = -d$; it is connected if and only if $d = \lambda_1 > \lambda_2$ (see eg. [12]). The expansion of a graph is frequently estimated in terms of λ_2 or the value $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$ considered above. Alon and Boppana (see Alon [1]) observed that for every infinite family G_1, G_2, \dots of d -regular graphs the second-largest eigenvalues satisfy

$$(7) \quad \liminf \lambda_2(G_i) \geq 2\sqrt{d-1}.$$

Although relatively sparse random graphs are well-expanding, the explicit construction of infinite sequences of sparse expanding graphs turned out to be difficult. Lubotzky, Phillips and Sarnak [22] achieved a breakthrough by constructing, for infinitely many values of d , infinite sequences of connected d -regular graphs that satisfy $|\lambda_i| \leq 2\sqrt{d-1}$ for all i with $\lambda_i \neq \pm d$, which is best possible by (7). They called graphs with this property *Ramanujan graphs*, because the algebraic construction of these graphs is related to a Ramanujan conjecture. Some of the sequences of Ramanujan graphs that they constructed are bipartite (thus $\lambda_n = -d$) and some are non-bipartite.

The latter graphs satisfy $\lambda \leq 2\sqrt{d-1}$ and hence they are f -connected for a linear function f by [Lemma 5.2](#) if $d \geq 3$.

Lubotzky, Phillips and Sarnak [22] observed that the graphs that they constructed have large girth and small independence number compared with the order of the graph. Combining [Lemma 5.2](#) with [Theorems 2.4 and 2.5](#) we get that non-bipartite Ramanujan graphs with $d \geq 3$ have a cycle of length linear in n . For larger d they even have factors and cycles of length close to n .

Corollary 5.3. *Let G be a non-bipartite d -regular Ramanujan graph of order n . If $d \geq 35$, then G contains a 1-factor (if n is even), a 2-factor, and a cycle of length at least $\frac{d}{d+2\sqrt{d-1}}n > \frac{3}{4}n$. If [Conjecture 2.3](#) is true, then G is hamiltonian.*

Proof. Since G is connected and non-bipartite we have $\lambda < d$, and hence $\lambda \leq 2\sqrt{d-1}$ as G is a Ramanujan graph. For $d \geq 35$ this implies $d/\lambda \geq d/(2\sqrt{d-1}) > 3$. By [Lemma 5.2](#), therefore, G is f -connected with $f(k) \geq \lceil (\frac{d}{\lambda} - 1)k \rceil \geq 2k + 1$. The assertions now follow from our [Theorems 3.2, 3.1 and 2.4](#). ■

Very recently, Krivelevich and Sudakov [20] proved that every d -regular graph with

$$\lambda = \frac{(\log \log n)^2}{1000(\log n \log \log n)}$$

of sufficiently large order n is hamiltonian. Compared to our [Theorem 2.1](#), which can only apply to graphs with $\Omega(n^{3/2})$ edges, their result applies to much sparser graphs with $\Omega(n \cdot g(n))$ edges, where $g(n)$ is a polylogarithmic function. Nevertheless, apart from the regularity requirement, their proof makes use of the fact that the number of edges joining any two large sets is not only bounded from below but also from above by λ . This is not the case for f -connected graphs.

Krivelevich and Sudakov [20] also offer the conjecture that there is a constant $c > 0$ such that every d -regular graphs with $\lambda < c \cdot d$ is hamiltonian, which would apply to graphs with linearly many edges. In view of [Lemma 5.2](#) the truth of our [Conjecture 2.2](#) would imply this conjecture.

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